# Stability Results for Advection-Diffusion Equations with Deterministic and Random Vector Fields

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# Outline

# Introduction: Advection-Diffusion Equations

# Advection-Diffusion Equations

Let $\mathrm{D} \subseteq \mathbb{R}^d$, and consider the continuity equation for a passive scalar $\theta(t,x)$ under the action of a vector field $u(t,x)$ with $\kappa \geq 0$,

$$\begin{cases} \partial_t \theta + \nabla \cdot (u\theta) = \kappa \Delta \theta & \text{in } (0,T) \times \mathrm{D}, \\ \theta(0,\cdot) = \theta^0 & \text{in } \mathrm{D}. \end{cases} \tag{AD}$$

If D has a boundary: $(u - \kappa \nabla \theta) \cdot n = 0 \quad \text{on } (0,T) \times \partial \mathrm{D}$.

Figure: Action of an alternating shear flow

# Advection-Diffusion Equations

Different viewpoint: follow the particle trajectories, given by the flow map

$$\begin{cases} dX_t &= u(t, X_t)dt + \sqrt{2\kappa}dB_t - n(X_t)dL_t, \\ X_0 &= \mathtt{id}, \end{cases} \tag{SDE}$$

where $(B_t)_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^d$, and $(L_t)_{t\geq 0}$ is a local time of the process $X_t$ that only activates when $X_t$ touches the boundary $\partial$D.

- Solutions to (AD) and (SDE) are related through **Feynman-Kac**:

$$\theta(t, \cdot) = \mathbb{E}[(X_t)_\# \theta^0].$$

**DiPerna-Lions setting**: $u \in L^1(W^{1,p})$ with $p > 1$, and $(\nabla \cdot u)^- \in L^1(L^\infty)$.

### Theorem [DiPerna, Lions (1989)]

For $\kappa = 0$, let $\theta^0 \in L^q$ with $1/p + 1/q \geq 1$. Then there exists a unique distributional solution $\theta \in L^\infty(L^q)$.

# Control over the Gradient of $\theta$

$\kappa > 0$ yields a control over the gradient of $\theta$:

**Theorem [Le Bris, Lions (2008)]**

If $\kappa > 0$, let $p = 2$ and $\theta^0 \in L^2 \cap L^\infty$. Then there exists a unique distributional solution $\theta \in L^\infty(L^2 \cap L^\infty) \cap L^2(\dot{H}^1)$.

For our estimates with $\kappa > 0$, we want $\nabla\theta$ to be controlled in $L^1(L^1)$. Consider initial data with **finite entropy**,

$$\int_D \theta^0 \log \theta^0 dx < \infty \quad \Rightarrow \quad \iint_{(0,T)\times D} |\nabla\theta| dxdt \lesssim \sqrt{\frac{T}{\kappa}}.$$

How to achieve finite entropy?

- **Bounded domain**: $\theta^0 \in L^q$, $q > 1$.
- **Unbounded domain**: $\theta^0 \in L^1 \cap L^q$, $q > 1$ and finite first moments.

# Optimal Transport Distances

Let $\mu, \nu \in L^1$ measures of equal mass, $\Pi(\mu, \nu)$ the set of all transport plans between them, and $c : [0, \infty) \to [0, \infty)$ a nondecreasing **cost function**. The optimal transport distance is defined via the minimisation problem

$$\mathcal{D}_c(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \iint_{\mathrm{D}\times\mathrm{D}} c(|x-y|)d\pi(x,y).$$

- $c(z)$ is a distance: $\mathcal{D}_c$ metrizes **weak convergence** of measures.
- $c(z)$ is concave: the OT problem admits a dual formulation,

$$\mathcal{D}_c(\mu, \nu) = \sup_{|\xi(x)-\xi(y)|\leq c(|x-y|)} \int_{\mathrm{D}} \xi(z)d(\mu-\nu)(z).$$

We consider a logarithmic cost function,

$$c(z) = \log\left(\frac{z}{\delta}+1\right) \quad \text{with} \quad \delta > 0.$$

# Stability for Distributional Solutions

# Stability in the DiPerna-Lions Setting

In the DiPerna-Lions setting, we find bounds on the the distance between two distributional solutions to (AD) given by different data.

- $u \in L^1(W^{1,p})$, $(\nabla \cdot u)^- \in L^1(L^\infty)$,
- $\theta^0 \in L^1 \cap L^q$ and finite first moments.

**Theorem 1 [NF, Schlichting, Seis]**

Let $\theta_1, \theta_2 \in L^\infty(L^q) \cap L^1(W^{1,1})$ be the unique solutions to (AD) defined by $(u_1, \kappa_1, \theta_1^0)$ and $(u_2, \kappa_2, \theta_2^0)$ respectively. Then we find the following stability estimate,

$$\sup_{t \in (0,T)} \mathcal{D}_\delta(\theta_1, \theta_2)(t) \lesssim 1 + \mathcal{D}_\delta(\theta_1^0, \theta_2^0) + \frac{\|u_1 - u_2\|_{L^1(L^p)}}{\delta} + \frac{|\kappa_1 - \kappa_2|}{\delta},$$

for every $\delta > 0$.

# Optimality of the Estimate and Zero-Diffusivity Limit

**Rate of convergence**: smallest $\delta = \delta_n$ for which the RHS is finite,

1. Initial data: $\mathcal{D}_{\delta_n}(\theta^0, \theta_n^0) \sim 1$. Optimal for weak convergence.
2. Vector field: $\|u - u_n\|_{L^1(L^p)} \sim \delta_n$. Optimal if $\kappa = 0$.
3. Diffusivity constant: $|\kappa - \kappa_n| \sim \delta_n$. Best known rate. Optimal?

$$\frac{t|\kappa_1 - \kappa_2|}{\sqrt{\kappa_1} + \sqrt{\kappa_2}} \lesssim W_1(\theta_1, \theta_2)(t), \quad \theta_1, \theta_2 \text{ heat kernels.}$$

**The zero-diffusivity limit**: let $u_1 = u_2$ and $\theta_1^0 = \theta_2^0$,

$$\sup_{t \in (0,T)} \mathcal{D}_\delta(\theta_1, \theta_2)(t) \lesssim 1 + \frac{|\kappa_1 - \kappa_2| \|\nabla \theta_2\|_{L^1(L^1)}}{\delta} \lesssim 1 + \frac{|\kappa_1 - \kappa_2|}{\delta} \sqrt{\frac{T}{\kappa_2}}.$$

- Seis (2018).
  In the limit $\kappa \to 0$, the optimal rate is $\mathcal{D}_\delta(\theta, \theta^\kappa)(t) \lesssim 1 + \sqrt{t\kappa}/\delta$.

# Stability out of the DiPerna-Lions Setting

We prove well-posedness of the Cauchy problem (AD) out of the DiPerna-Lions setting, see Bouchut, Crippa (2013).

- $\nabla u = K * \omega$ where $\omega \in L^1(L^1)$ and $K$ is a singular integral kernel,
- $\theta^0 \in L^1 \cap L^\infty$ mean free.

**Theorem 2 [NF, Schlichting, Seis]**

The Cauchy problem (AD) has a unique distributional solution with

$$\theta \in L^\infty(L^1 \cap L^\infty) \quad \text{and} \quad \nabla\theta \in L^1(L^1).$$

Uniqueness is a byproduct of the estimate: $\forall \varepsilon > 0$, $\exists C_\varepsilon > 0$ such that

$$\sup_{t \in (0,T)} \mathcal{D}_\delta(\theta, 0)(t) \lesssim \mathcal{D}_\delta(\theta^0, 0) + \varepsilon \left(1 + \log \frac{\|u\|_{L^{p,\infty}}}{\varepsilon\delta}\right) + C_\varepsilon,$$

for all $\delta > 0$.

# Stability for the Implicit Finite Volume Scheme

# Unstructured Meshes

Let $\mathrm{D} \subset \mathbb{R}^d$ be bounded, let $\partial\mathrm{D}$ be $C^{1,1}$, and consider

- $\{K\}_{K\in\mathcal{T}} \subset \mathrm{D}$ a tessellation with closed, polygonal cells;
- $h = \max \operatorname{diam} K$ size of the mesh.

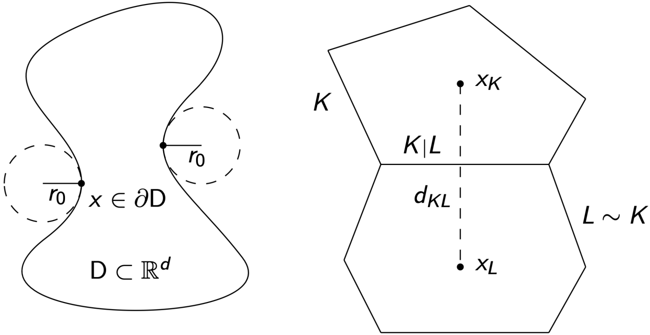


Figure: Exterior ball condition and an example of control cell.

# The Implicit Finite Volume Scheme

Let $\tau > 0$ be the time step.

- Initial datum averaged on every cell $\theta_K^0 = \fint_K \theta^0 dx$.
- Discrete normal velocity from control cell $K$ to neighboring $L$,

$$u_{KL}^n = \fint_{n\tau}^{(n+1)\tau} \fint_{K|L} u \cdot n_{KL} dH^{d-1} dt.$$

Then the finite volume scheme is given by

$$\frac{\theta_K^{n+1} - \theta_K^n}{\tau} + \sum_{L \sim K} \frac{|K|L|}{|K|} \left( u_{KL}^{n+} \theta_K^{n+1} - u_{KL}^{n-} \theta_L^{n+1} + \kappa \frac{\theta_K^{n+1} - \theta_L^{n+1}}{d_{KL}} \right) = 0.$$

The approximate solution $\theta_{\tau h}$ is defined by

$$\theta_{\tau h}(t,x) = \theta_K^n \quad \text{a.e. } (t,x) \in [n\tau, (n+1)\tau) \times K. \tag{FV}$$

# Stability for the Implicit Finite Volume Scheme

We study the converegence of the approximate solution towards the distributional solution on the DiPerna-Lions setting:

- $u \in L^1(W^{1,p})$ with $p \in (1, \infty]$, $(\nabla \cdot u)^- \in L^1(L^\infty)$;
- $\theta^0 \in L^q$ with $q \in (1, \infty]$ and $1/p + 1/q \leq 1$.

In addition we assume: $u \in L^\infty((0, T) \times D)$.

**Theorem 3 [NF, Schlichting]**

Let $\theta \in L^\infty(L^q) \cap L^1(W^{1,1})$ be the unique distributional solutions to (AD) and $\theta_{\tau h}$ the unique approximate solution given by (FV). Then, for $\tau > 0$ small enough, there holds

$$\sup_{t \in (0,T)} \mathcal{D}_\delta(\theta, \theta_{\tau h})(t) \lesssim 1 + \frac{h}{\delta} + \frac{\sqrt{\tau T} \|u\|_\infty}{\delta} + \frac{\sqrt{\tau \kappa}}{\delta},$$

for every $\delta > 0$.

# Numerical Diffusion and Optimality

$$\sup_{t\in(0,T)} \mathcal{D}_\delta(\theta,\theta_{\tau h})(t) \lesssim 1 + \frac{h}{\delta}\min\left\{\frac{1}{\sqrt{h}},\frac{1}{\sqrt{\kappa}}\right\} + \frac{\sqrt{\tau}}{\delta}$$

- Guo, Stynes (1997), Droniou (2002).
  Rate of convergence with $\kappa > 0$ and smooth vector field: $h$.
- Schlichting, Seis (2018).
  Rate of convergence with $\kappa = 0$ in DiPerna-Lions: $\sqrt{h}$.

How do we improve the rate of convergence? **BV estimates**:

$$\tau \sum_n \sum_K \sum_{L\sim K} |K|L||\theta_K^{n+1} - \theta_L^{n+1}| \lesssim \min\left\{\frac{1}{\sqrt{h}},\frac{1}{\sqrt{\kappa}}\right\}.$$

The discretization of $\mathrm{D} \subset \mathbb{R}^d$ generates numerical diffusion that heuristically corresponds to a second diffusion with coefficient $h > 0$,

$$\partial_t\theta + u\cdot\nabla\theta = (\kappa + h)\Delta\theta.$$

# Ergodicity and Mixing with Random Vector Fields

# Transport by Random Vector Fields

Let $\kappa = 0$ and consider the transport of the passive scalar $\theta(t,x)$ by a divergence free vector field $u(t,x)$,

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0 & \text{in } (0,\infty) \times \mathrm{D}, \\ \theta(0,\cdot) = \theta^0 & \text{in } \mathrm{D}. \end{cases} \tag{T}$$

If D has a boundary, we impose $u \cdot n = 0$ on $(0,\infty) \times \partial \mathrm{D}$. Then,

- the total mass of $\theta(t,\cdot)$ is conserved;
- $\|\theta(t)\|_{L^p} = \|\theta^0\|_{L^p}$ for all $p \in [1,\infty]$.

**Question**: Can we find examples of vector fields $u$ that make

$$\theta(t,\cdot) \rightharpoonup \int_{\mathrm{D}} \theta^0 dx \overset{\text{WLOG}}{=} 0$$

as $t \to \infty$ in some sense? How fast can the convergence be?

# Random Vector Fields

Let $\mathrm{D}, \mathrm{D}' \subset \mathbb{R}^d$ be bounded, we choose vector fields the form

$$u = u(x, y) \quad \text{with } (x, y) \in \mathrm{D} \times \mathrm{D}'.$$

In $\mathrm{D}'$, define a Brownian motion $(Y_t)_{t \geq 0}$ of *intensity* $\nu > 0$,

$$dY_t = \sqrt{2\nu} dB_t - n(Y_t) dL_t, \quad Y_0 = \mathtt{id}.$$

The **random vector field** in (T) is defined by $u(x, Y_t)$, an depends implicitly on the noise realisation and initial point $Y_0 \in \mathrm{D}'$.

- We look for examples of vector fields $u(\cdot, Y_t)$ that make the passive scalar $\theta(t, \cdot)$ being **exponentially ergodic**,

$$\|\mathbb{E}\theta(t)\|_{L^2} \to 0 \quad \text{exponentially fast.}$$

# Ergodicity and Annealed Mixing

- We obtain a coupled SDE in $\mathrm{D} \times \mathrm{D}'$:

$$\begin{pmatrix} dX_t \\ dY_t \end{pmatrix} = \begin{pmatrix} u(X_t, Y_t) \\ 0 \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sqrt{2\nu} \end{pmatrix} dB_t - \begin{pmatrix} 0 \\ n(Y_t) \end{pmatrix} dL_t$$

To study the long-time behaviour of this system there are two perspectives.

1. **ODE**: random dynamics + Markov process

$$P_t(x, A) = \mathbb{P}[X_t \in A \mid X_0 = x], \quad A \in \mathcal{B}(\mathrm{D}),$$

   see Bedrossian, Blumenthal, Punshon-Smith (2018, 2019, 2020).

2. **PDE**: we can use Feynman-Kac to transform the SDE into a PDE

$$\begin{cases} \partial_t f + u(x,y) \cdot \nabla_x f = \nu \Delta_y f, & \text{in } \mathrm{D} \times \mathrm{D}', \\ n_y \cdot \nabla_y f = 0, & \text{on } \in \mathrm{D} \times \partial\mathrm{D}', \end{cases} \tag{PDE}$$

   where $f(t,x,y) = \mathbb{E}[(X_t, Y_t)_\# f^0(x,y)]$.

# Hypocoercivity and Ergodicity

Let $u \in C^2(\mathrm{D} \times \mathrm{D}')$, and $f(0,x,y) = \theta^0(x)\rho(y)$ with

- $\mathrm{law}\, Y_0 = \rho \in (\mathcal{P} \cap W^{1,\infty})(\mathrm{D}')$;
- $\theta^0$ mean free, and $\theta^0 \in H^1(\mathrm{D})$.

**Theorem 4 [NF, Schlichting, Seis]**

If $\exists \gamma > 0$ such that $\|f(t)\|_{H^1} \lesssim e^{-\gamma t}$, then $\|\mathbb{E}\theta(t)\|_{L^2} \lesssim e^{-\gamma t}$.

Namely, we found a *sufficient condition* for exponential ergodicity:

**hypocoercivity** of $\mathcal{L} = u(x,y) \cdot \nabla_x - \nu \Delta_y$ (+ BC), see Villani (2009).

- **Example 1**: Shear flows with random phases in $\mathbb{T}^2$,

$$u_{\mathrm{shear}}(x,y) = \begin{pmatrix} \sin(x_2 + y_1) \\ \sin(x_1 + y_2) \end{pmatrix}, \quad (x,y) \in \mathbb{T}^2 \times \mathbb{T}^2.$$

# Examples of Ergodicity with a Random Vector Field

- **Example 2**: Randomly moving vortex in $B_1 \subset \mathbb{R}^2$,

$$u_{\text{vortex}}(x, y) = -\frac{e^{-2\pi\psi(x,y)}}{1 - |y|^2} \nabla_x^{\perp} \psi(x, y), \quad (x, y) \in B_1 \times B_r,$$

where $r < 1$ and $\psi(x, y)$ is the streamfunction of a point vortex in $y$.

**Theorem 5 [NF, Schlichting, Seis]**

Let $\nu \gg 1$ be sufficiently large, and $f$ be a solution to (PDE).

1. Given the vector field $u_{\text{shear}}(\cdot, Y_t)$, $\exists \alpha > 0$ such that

$$\|f(t)\|_{H^1(\mathbb{T}^2 \times \mathbb{T}^2)} \lesssim \|f^0\|_{H^1(\mathbb{T}^2 \times \mathbb{T}^2)} e^{-\alpha t}, \quad \forall t \geq 0.$$

2. Given the vector field $u_{\text{vortex}}(\cdot, Y_t)$, $\exists \beta > 0$ such that

$$\|f(t)\|_{H^1_\delta(B_1 \times B_r)} \lesssim \|f^0\|_{H^1_\delta(B_1 \times B_r)} e^{-\beta t}, \quad \forall t \geq 0.$$

# Examples of Ergodicity with a Random Vector Field

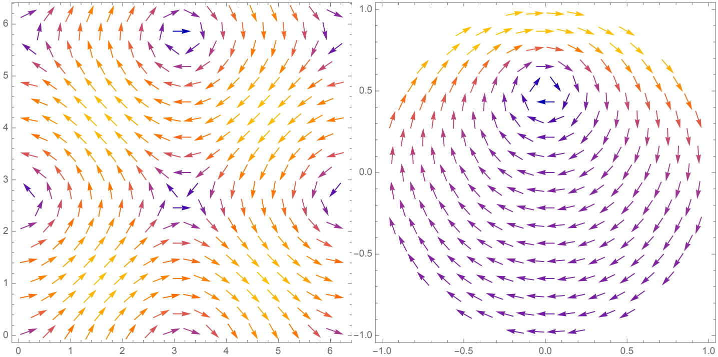


Figure: Streamlines of the two examples of vector fields with $y = (0, 0.5)$.

# Some Key References


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